

Efficient Entry in Competing Auctions*

James Albrecht (Georgetown University)

Pieter A. Gautier (VU Amsterdam)

Susan Vroman (Georgetown University)

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Abstract

In this paper, we demonstrate the efficiency of seller entry in a model of competing auctions in which we allow for both buyer and seller heterogeneity. This generalizes existing efficiency results in the competitive search literature by simultaneously allowing for nonrival (many-on-one) meetings and private information.

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In this paper, we consider the efficiency of seller entry into a large market in which sellers post and commit to terms of trade and buyers then allocate themselves across sellers. Buyers differ in their valuations for the good that is being sold in this market, and sellers are heterogeneous with respect to their reservation values. In this environment, a second-price auction is a simple mechanism that allows the buyers who interact with a particular seller to compete for that seller's good. The competing auctions literature, e.g., McAfee (1993) and Peters and Severinov (1997), considers the characteristics of these auctions as the market gets large, and it is now well understood that competition leads each seller to post an efficient mechanism, e.g., a second-price auction with reserve price equal to seller reservation value.

The implications of a large market for seller entry are less well understood in this environment. When goods are auctioned, buyers extract an information rent from sellers. One might therefore expect that the equilibrium level of seller entry into the market would be below the level that a social planner would choose. There is, however, a counterbalancing force, namely, that when a seller enters the market, that seller "steals" buyers from the sellers who were already there and so potentially reduces the surpluses associated with these sellers. Our main result is that in a large market when sellers are free to choose their preferred mechanisms, these two effects – information rent versus business stealing – exactly offset each other, leading to the socially efficient level of seller entry. In addition, we show that the equilibrium and social planner allocation of buyers across seller types coincide.

The model we present is one of competitive search. Sellers compete for buyers by posting and committing to selling mechanisms, and buyers direct their search based on these posted mechanisms. Moen (1997) and Shimer (1996) demonstrate the efficiency of entry in a model of competitive search that has two special features. First, meetings between buyers and sellers are one-on-one, that is, each seller interacts with at most one buyer. Second, there is complete information in the sense that once a buyer and seller meet private information is not an issue. We generalize this literature on efficient entry in competitive search in two directions. First, we allow for nonrival (many-on-one) meetings; that is, the fact that one or more buyers choose to visit a seller does not make

it more difficult for any other buyer to visit that seller.¹ Second, we allow for private information, which we model by assuming that sellers do not know how much the buyers at their auctions value the good.² With a one-on-one meeting technology and complete information, the only relevant mechanism for selling a good is price posting. With many-on-one meetings and asymmetric information, second-price auctions rather than wage posting constitute an optimal mechanism. Thus, we have a model of competing auctions.

1 Model

We consider a market with a double continuum of buyers and sellers. The measure of buyers is B , and we normalize the measure of sellers that could potentially enter the market to one. In this market, each buyer wants to purchase one unit; each seller has one unit of the good for sale. Every seller posts and commits to a selling mechanism, and each buyer, after observing all posted mechanisms, chooses one seller to interact with. The meeting technology is nonrival. Finally, as is standard in the competitive search literature, we assume that buyers cannot coordinate their visiting strategies.

We assume that buyer valuations for the good are distributed as $X \sim F(x)$, with corresponding density, $f(x)$, and we normalize the range of X to $[0, 1]$. Buyer valuations are independently and identically distributed and are private information; i.e., we are considering a model of “independent private values.” In our baseline case, we assume that buyers learn their valuations *ex post*, that is, only after choosing a particular seller.³ Sellers are *ex ante* heterogeneous with reservation values distributed as $S \sim G(s)$, with corresponding density, $g(s)$. Again, we normalize the range of the random variable to $[0, 1]$. We interpret seller type as motivation – the sellers with high reservation values are “relaxed” about selling their good while those with reservation values close to zero are “desperate” (or, as is often seen in housing ads, “motivated”). Finally, we describe the

¹Nonrival is the terminology used in the search literature for such meeting technologies. See Eeckhout and Kircher (2010).

²Efficiency of seller entry in competitive search with private information was shown by Albrecht and Jovanovic (1986) for one-on-one meetings using a particular matching function.

³We refer to our baseline model as the *ex post* case. We consider the *ex ante* case in which buyers know their valuations before choosing which seller to visit in Section 3.

allocation of buyers across the different seller types by a function $\theta(s)$, which gives the expected number of buyers per seller of type s .

We model the interaction of buyers and sellers as a one-shot game with the following stages:

1. Each type s seller decides whether or not to pay a cost A to enter the market. A type s seller who does not enter retains the reservation value s .
2. Each seller who enters the market posts and commits to a mechanism to sell one unit of the good.
3. Each buyer, after observing all posted mechanisms, chooses a seller.
4. Given the allocation of buyers to sellers, goods are allocated and payments are made according to the posted mechanisms.

The payoff for a type s seller consists of any payments received according to the posted mechanism plus s if the good is retained. The payoff for a buyer with valuation x equals x if the good is purchased and zero otherwise less any payments specified by the posted mechanism.

Definition 1 *An equilibrium for this game is a choice of a selling mechanism by each seller, an allocation of buyers across sellers, $\theta(s)$, and a cutoff seller type, s^* , such that (i) each seller's mechanism maximizes the seller's expected payoff given the mechanisms posted by the other sellers in the market and the rule that buyers follow to allocate themselves across sellers, (ii) each buyer chooses a probability distribution over seller types that maximizes the buyer's expected payoff given the mechanisms posted in the market and the allocation rule followed by the other buyers, and (iii) the payoff that seller s^* expects on entering the market equals $A + s^*$.*

In the independent private values environment that we consider, a second-price auction with reserve price equal to seller reservation value is an efficient mechanism. It guarantees

that the good is sold whenever at least one buyer has a valuation above the seller's reservation value and it also guarantees that, if the good is sold, it goes to the buyer with the highest valuation. Competition ensures that sellers post efficient mechanisms. The intuition can be seen in Levin and Smith (1994). They consider a single seller offering a second-price auction with reserve price r who faces a fixed number of potential buyers, each with a common outside option. Their result, that endogenizing buyer entry drives r to the seller's reservation value, even though there is only one seller in the market, is generated by the fixed outside option. Each buyer who participates in the auction must get an expected payoff equal to this outside option. It is therefore in the seller's interest to post an efficient mechanism. Posting an inefficient mechanism cannot reduce the expected payoff per buyer that the seller has to pay and therefore can only reduce the seller's expected payoff. The same result holds in the competing auctions environment (Albrecht, Gautier and Vroman 2012) with the difference that the outside option – now equal to “market utility,” i.e., the payoff that a buyer can expect to get elsewhere in the market – is endogenous.⁴

Levin and Smith (1994) also show that it is not in the seller's interest to post an entry fee, and the same result can be derived in the competing auctions framework using the approach of Albrecht et al. (2012). The fact that buyer entry fees are zero and reserve prices equal reservation values in a large market means that not only is efficiency ensured in the competing auctions equilibrium but also that the seller's expected payoff is limited to the expectation of the second highest valuation among buyers participating in the auction. This result is related to one derived in Gorbenko and Malenko (2011). They consider competition in “securities auctions” in which sellers auction off the right to develop projects for a combination of cash and a share of the profits. Getting buyers to pledge a profit share is a way for sellers to recapture some of the information rent associated with buyer private information. Proposition 4 in Gorbenko and Malenko (2011) shows that as the number of buyers and sellers in the market gets large, all-cash auctions are posted in the competitive search equilibrium. That is, as the market gets large,

⁴Of course, this efficiency result requires a large market. In a market with a small number of sellers, the terms of trade posted by any one seller affect market utility.

competition prevents sellers from “clawing back” any part of the information rent.⁵

2 Efficient Seller Entry

To discuss the efficiency of seller entry, some notation is useful. Consider a type s seller who posts a second-price auction with reserve price r , which generates an expected buyer arrival rate of $\theta(r)$. Let $\Pi(r, \theta(r); s)$ denote this seller’s expected payoff. Similarly, let $V(r, \theta(r))$ denote the expected payoff for a buyer who participates in an auction with reserve price r .⁶ This seller sets a reserve price to maximize $\Pi(r, \theta(r); s)$ subject to the constraint that $V(r, \theta(r)) = \bar{V}$, where \bar{V} is market utility. Competition leads the seller to choose $r = s$.

The surplus associated with a second-price auction posted by this seller with reserve price equal to reservation value is the expected maximum of s and the highest valuation drawn by buyers who interact with this seller. Denoting this surplus by $\Lambda(s, \theta(s); s)$,⁷ we have

$$\Lambda(s, \theta(s); s) = \Pi(s, \theta(s); s) + \theta(s)V(s, \theta(s)). \quad (1)$$

The surplus associated with this auction equals the seller’s expected payoff plus the expected number of buyers participating in the auction, $\theta(s)$, times the expected payoff per buyer. Because the mechanism is efficient, no surplus is “left on the table.”

Our proof that seller entry into this market is efficient relies on the fact that

$$\frac{\partial \Pi(s, \theta(s); s)}{\partial \theta(s)} + \theta(s) \frac{\partial V(s, \theta(s))}{\partial \theta(s)} = 0. \quad (2)$$

That is, if the expected number of buyers per type s seller increases, the increase in this seller’s expected payoff is just offset by the expected decrease in payoffs across the seller’s

⁵It is worth noting that the same logic – competition leads sellers to post efficient mechanisms – applies with a rival meeting technology. However, the efficient mechanism in this case does include an entry fee. See Albrecht and Jovanovic (1986) and Faig and Jerez (2005).

⁶The seller’s expected payoff depends on the reserve price posted, the expected arrival rate of buyers *and* on the seller’s type, s , since s is retained if the good is not sold. The buyer’s expected payoff only depends on the posted reserve price and the expected arrival rate of other buyers.

⁷In Appendix A.1, we derive expressions for $\Pi(s, \theta(s); s)$, $V(s, \theta(s))$ and $\Lambda(s, \theta(s); s)$.

“incumbent” buyers.⁸ The intuition for equation (2) can most easily be understood in a single-seller setting.

Consider a type s seller posting a second-price auction with reserve price s . Suppose n buyers initially participate in the auction. Now suppose an $(n+1)^{st}$ buyer is added. Total surplus only increases if the new buyer wins the auction. In this case, the buyer gets his or her valuation minus the maximum of s and the highest valuation across the original n buyers; that is, the marginal contribution to the surplus associated with the auction equals the new buyer’s payoff. This means, as equation (2) indicates, that the entry of a new buyer does not change the sum of the payoffs to the seller and to the original n buyers.

We can apply this intuition to the large market setting. Suppose a new type s seller enters the market, posting a second-price auction with reserve price s . This seller can expect to attract $\theta(s)$ buyers away from sellers who were already in the market. Since the market is large, the probability that two or more of these buyers come from any one auction that was already in the market is negligible. The new seller’s auction creates a surplus equal to the highest valuation drawn by the buyers it attracts (or s if it attracts no buyers drawing $x \geq s$). This seller’s payoff equals the second-highest valuation (or s if the auction attracts no buyers or only one buyer who draws $x \geq s$). The difference between the surplus associated with the auction and the seller’s payoff is the information rent.

Now consider the buyers (if any) who are attracted to the new auction. If the new seller hadn’t entered the market, each of these buyers would have participated in some other seller’s auction. The buyers who choose the new auction decrease the surpluses associated with the auctions they leave. In each of these auctions, there is a decrease in the expected seller payoff and an increase in the expected payoffs across the remaining

⁸In Albrecht et al. (2012), we consider the problem of maximizing $\Pi(r, \theta(r); 0)$ subject to $V(r, \theta(r)) = \bar{V}$ using the expressions for $\Pi(\cdot)$ and $V(\cdot)$ derived in Peters and Severinov (1997). We verify equation (2) for the case of $s = 0$ in that paper by showing that this equation is one of the first-order conditions for the seller’s constrained maximization problem. In Appendix A.1, we present an alternative approach to verifying this equation; namely, we use the result that competition leads sellers to post efficient mechanisms ($r = s$) to develop simple expressions for the terms in equation (1).

buyers, but these two effects cancel, so the decrease in surplus associated with each of these auctions equals the payoff the departing buyer could have expected there, namely, the market utility, \bar{V} . Similarly, each of these buyers increases the surplus associated with the new seller's auction. Each buyer who participates in the new seller's auction increases that seller's expected payoff but decreases the expected payoffs for any other buyers who visit the new seller. Again, these two effects are exactly offsetting. That is, the increase in surplus at the new seller's auction generated by adding one more buyer equals that buyer's expected payoff at that auction. Since each of the buyers who participates in the new seller's auction anticipates competing with $\theta(s)$ other buyers at that auction, each buyer's expected payoff at the new auction is $V(s, \theta(s))$. In equilibrium, $V(s, \theta(s)) = \bar{V}$, so the decrease in surplus at an auction that loses a buyer equals the increase in surplus that buyer adds to the entrant's auction. The decrease in surplus at auctions that lose buyers is the business-stealing effect, $\theta(s)\bar{V}$, and this equals the sum of the contributions these buyers make to the new seller's auction, $\theta(s)V(s, \theta(s))$. From equation (1), we have $\Pi(s, \theta(s); s) = \Lambda(s, \theta(s); s) - \theta(s)V(s, \theta(s))$; that is, the expected payoff for a new entrant of type s equals the increase in surplus generated by the auction net of the decrease in surplus caused by the departure of buyers from auctions that were already in the market. Thus, the information-rent and business-stealing effects are exactly offsetting.

We now consider the social planner problem. The social planner chooses a level of seller entry and an allocation of buyers across seller types to maximize total surplus in the market net of entry and opportunity cost. That is, the social planner chooses a cutoff seller type, s^* , – all sellers of type $s \leq s^*$ enter the market; all types $s > s^*$ stay out – and a buyer-seller ratio for each seller type, $\theta(s)$, to maximize

$$\int_0^{s^*} (\Lambda(s, \theta(s); s) - (A + s))g(s)ds.$$

This maximization is carried out subject to

$$\int_0^{s^*} \theta(s)g(s)ds = B;$$

that is, the allocation of buyers across seller types has to satisfy an adding-up constraint.

Proposition 1 *The social planner values of $\theta(s)$ and s^* equal the free-entry equilibrium values.*

Proof. The Lagrangean for the social planner problem is

$$\int_0^{s^*} (\Lambda(s, \theta(s); s) - (A + s)) g(s) ds + \lambda \left(B - \int_0^{s^*} \theta(s) g(s) ds \right)$$

with first-order conditions

$$\Lambda(s^*, \theta(s^*); s^*) - (A + s^*) - \lambda \theta(s^*) = 0 \quad (3)$$

$$\frac{\partial \Lambda(s, \theta(s); s)}{\partial \theta(s)} - \lambda = 0 \quad \forall s \in [0, s^*] \quad (4)$$

$$B - \int_0^{s^*} \theta(s) g(s) ds = 0 \quad (5)$$

Using equations (1) and (2), equation (4) can be rewritten as

$$V(s, \theta(s)) = \lambda \quad \forall s \in [0, s^*]. \quad (6)$$

That is, the social planner wants buyers to allocate themselves across sellers so that the values of visiting the various seller types in the market are equalized. This same condition holds in equilibrium since buyers must be indifferent across seller types.

Next, consider equation (3). This can be rewritten as

$$\Pi(s^*, \theta(s^*); s^*) - (A + s^*) + \theta(s^*) (V(s^*, \theta(s^*))) - \lambda = 0.$$

From equation (6), we have $V(s^*, \theta(s^*)) = \lambda$, so equation (3) requires $\Pi(s^*, \theta(s^*); s^*) = A + s^*$. That is, the marginal entrant should be just indifferent with respect to entering the market. Again, the same condition holds in equilibrium with free entry.⁹ ■

3 The Ex Ante Case

In this section, we consider the case in which buyers know their valuations *ex ante*, i.e., before choosing which seller to visit. We begin by describing the equilibrium in competing

⁹In our related working paper, Albrecht et al. (2013), we prove a similar result for the case of two seller types.

auctions for this case taking the distribution of sellers in the market as given. Then we discuss the efficiency of seller entry and of the allocation of buyers across seller types. The details are given in Appendix A.2.

Competition in selling mechanisms in the *ex ante* case was first considered by McAfee (1993). Sellers post efficient mechanisms, as in the *ex post* case, but, interestingly, the reasoning is different. McAfee (1993) makes the following argument to show that when sellers all have reservation value 0, each posts a second-price auction with a zero reserve price.¹⁰ Suppose all sellers post positive reserve prices, so buyers with low valuations are shut out of the market. Now consider the seller posting the lowest reserve price, say $r' > 0$. If this seller deviates to $r = 0$, he or she captures the entire market between zero and r' . What is more interesting is that the deviant seller doesn't lose any buyers (in expectation) with valuations above r' . Buyers with valuations above r' don't care about competition from buyers with valuations below r' . More precisely, a buyer with valuation $x > r'$ knows that r' is the lowest possible price irrespective of whether the deviant seller is posting $r = r'$ or $r = 0$ because all buyers with $x \leq r'$ visit this seller. Thus, buyers with valuations above r' continue to allocate themselves across all sellers (including the deviant) exactly as they would have absent the deviation. This argument further implies that no seller (regardless of what reserve prices are set by the other sellers) wants to set a positive reserve price. Thus, competition drives the equilibrium reserve price to the common seller reservation value of zero in the *ex ante* case.

When sellers are heterogeneous with respect to their reservation values, the equilibrium is more complicated. It is simplest to start with the intuition for the case of two seller types. Suppose there is a mass m_1 of sellers with reservation value s_1 and a mass m_2 of sellers with reservation value $s_2 > s_1$. Competition leads the sellers to post reserve prices equal to their reservation values.

Buyers then sort themselves based on their valuations. Buyers with valuations below s_1 are shut out of the market. Similarly, it is pointless for a buyer with valuation $x < s_2$ to visit a seller posting s_2 . Essentially the same is true for buyers with valuations that are slightly higher than s_2 – these buyers are also better off visiting a seller posting s_1 , even

¹⁰Our treatment follows Peters and Severinov (1997) and Peters (2013).

though there may be more competition from other buyers there. Let x^* be the lowest valuation that is consistent with indifference between visiting the two seller types. Buyers with $x > x^*$ are also indifferent between the two seller types and visit all sellers with equal probability. Why do buyers with sufficiently high valuations randomize across *all* sellers? To understand this, consider the buyer of type x^* . So long as buyers with higher valuations randomize their visits across all sellers, the buyer with valuation x^* is equally likely to be the high bidder in an auction with reserve price s_2 as in an auction with reserve price s_1 . The threshold value x^* is then determined by equating the expected payoffs in the two auctions, conditional on being the high bidder. Next, consider a buyer with valuation $x' > x^*$. If all buyers with even higher valuations randomize their visits across all sellers, then the buyer with valuation x' is equally likely to be the high bidder in either seller type's auction. Finally, given that expected payoffs conditional on winning are the same for the buyer with valuation x^* and given that buyers with valuations between x^* and x' randomize their visits across all sellers, the expected payoff, conditional on winning, for the buyer with valuation x' is the same in an auction with reserve price s_1 as one with reserve price s_2 .

Continuing with two seller types, the next step is to allow for free entry. That is, m_2 is endogenized in free-entry equilibrium. In Appendix A.2, we consider the two-seller type case in detail. We characterize the equilibrium allocation of buyers across the two seller types and the distribution of seller types in the market in terms of x^* and m_2 , and we show that these values are the same as those that would be chosen by a social planner. In short, we show that the equilibrium level of seller entry and the allocation of buyers across the two seller types are efficient in competitive search equilibrium. We then generalize to allow for an arbitrary, discrete number of seller types. Finally, by taking the appropriate limit, we consider the model with a continuum of seller types. The equilibrium allocation of buyers across sellers, that is, the expected number of buyers choosing each seller type and the distribution of buyer types across sellers, is described by a function $x^*(s)$ that gives the threshold valuation associated with each seller type, and the marginal seller type is determined by free entry.

The intuition for the efficient allocation of buyers across seller types and efficient seller

entry in a model of competing auctions when buyers know their valuations *ex ante* is not so different from the intuition given above in the *ex post* case. Consider again the model with two seller types. From the planner’s perspective, buyers with valuations below x^* contribute less to auctions posted by sellers of type s_2 than they do to auctions posted by the other seller type. In equilibrium, the contribution these buyers would make to each auction type equals their expected payoffs, so these buyers have the correct incentives to direct their search efficiently. That is, the market uses the *ex ante* information efficiently in the sense that low valuation buyers do not wastefully visit high seller types. Similarly, buyers who find it worthwhile to search both seller types should do so in such a way that the surplus they add by participating in an auction with reserve price s_1 is the same as what they add by participating in an auction with reserve price s_2 . Again, because these buyers allocate themselves between the two auction types to equalize their expected payoffs, equilibrium incentives give the socially optimal allocation of buyers across the two seller types. In terms of seller entry, the marginal entrant has reservation value s_2 . Only high-valuation buyers will consider visiting this entrant. In equilibrium, the expected payoff for high-valuation buyers is the same whether they visit sellers posting s_1 or s_2 . Thus, whichever auction type the marginal entrant draws buyers away from, the business-stealing effect is the same. Exactly as in the *ex post* case, the expected contribution that a buyer makes to the new entrant’s auction equals the expected loss in surplus that his or her “departure” generates in the auction in which he or she would otherwise have participated. Once again, the information-rent and business-stealing effects are exactly offsetting, giving the correct incentives for entry.

4 Conclusion

The constrained efficiency of competitive search equilibrium is well understood when meetings between buyers and sellers take place on a one-on-one basis with complete information. However, in many situations, e.g., in standard auction settings, it is more appropriate to assume a nonrival meeting technology, i.e., many-on-one meetings. These are situations in which buyers differ in terms of how much they value the good that is

being offered for sale and in which these valuations are private information.

Our main contribution in this paper is to show that the allocation of buyers across seller types and the level of seller entry are efficient in competitive search equilibrium with nonrival meetings. To get efficient entry, sellers who could potentially enter the market need the correct incentives. The payoff expected by the marginal entrant should equal his or her expected contribution to market surplus. The expected contribution to market surplus from this auction is the maximum of the seller's reservation value, s , and the expectation of the highest valuation drawn by buyers who participate in that auction, while the seller's expected payoff is the expectation of the maximum of s and the second-highest valuation across buyers in the auction. The difference between what the winning buyer expects to get and what the seller expects to receive is an information rent, and the existence of this information rent makes it seem that there will not be enough seller entry. This, however, neglects the fact that the buyers who participate in the marginal entrant's auction would have participated in some other seller's auction had the last seller not entered the market. That is, the entry of the marginal seller creates a business-stealing externality. Our main contribution, therefore, can be understood as showing that the information-rent and business-stealing effects exactly offset each other in competitive search equilibrium, thus generating the efficient level of seller entry.

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A Appendix: For Online Publication

A.1 *Ex Post* Case

In this appendix, we verify equation (2) and derive expressions for $\Lambda(s, \theta(s); s)$, $\Pi(s, \theta(s); s)$, and $V(s, \theta(s))$. As noted in the text, in Albrecht et al. (2012), we analyzed the problem

$$\max_{r, \theta(r)} \Pi(r, \theta(r); 0) \text{ subject to } V(r, \theta(r)) = \bar{V}$$

using expressions for $\Pi(r, \theta(r); 0)$ and $V(r, \theta(r))$ that were derived in Peters and Severinov (1997). This is the problem of a seller of type $s = 0$ choosing a reserve price for a second-price auction who takes into account that the reserve price determines the expected arrival rate of buyers to the auction. In the process of showing that $r = 0$, that is, that the optimal reserve price equals the seller reservation value, we derived

$$\frac{\partial \Pi(0, \theta(0); 0)}{\partial \theta} + \theta(0) \frac{\partial V(0, \theta(0))}{\partial \theta} = 0$$

as one of the first-order conditions for the seller's constrained maximization problem. The same approach can be used to confirm equation (2),

$$\frac{\partial \Pi(s, \theta(s); s)}{\partial \theta} + \theta(s) \frac{\partial V(s, \theta(s))}{\partial \theta} = 0$$

for arbitrary s .

Here we present an alternative approach to verifying equation (2). We use the result that competition leads sellers to post efficient auctions; that is, sellers post reserve prices equal to reservation values. This gives equation (1) in the text, namely,

$$\Lambda(s, \theta(s); s) = \Pi(s, \theta(s); s) + \theta(s)V(s, \theta(s)).$$

Differentiating gives

$$\frac{\partial \Lambda(s, \theta(s); s)}{\partial \theta} = \frac{\partial \Pi(s, \theta(s); s)}{\partial \theta} + \theta(s) \frac{\partial V(s, \theta(s))}{\partial \theta} + V(s, \theta(s)),$$

and our proof of equation (2) consists of showing

$$\frac{\partial \Lambda(s, \theta(s); s)}{\partial \theta} = V(s, \theta(s)). \tag{7}$$

To do this, we develop expressions for $\Lambda(s, \theta(s); s)$ and $V(s, \theta(s))$. We also derive $\Pi(s, \theta(s); s)$ and verify that our expressions for seller and buyer expected payoffs are equivalent to the ones given in Peters and Severinov (1997).

We begin with an expression for $\Lambda(s, \theta(s); s)$, the expected surplus associated with the auction posted by seller s , which equals the expected maximum of s and the highest valuation greater than s drawn by one of the buyers. Suppose n buyers visit this seller and draw valuations of s or more. If $n = 0$, then $\Lambda(s, \theta(s); s) = s$. If $n \geq 1$, then, conditional on n , the expected surplus is

$$\int_s^1 x d \left(\frac{F(x) - F(s)}{1 - F(s)} \right)^n = 1 - \int_s^1 \left(\frac{F(x) - F(s)}{1 - F(s)} \right)^n dx,$$

that is, the expected maximum of the n draws that are greater than or equal to s .

The number of buyers visiting seller s who draw valuations of s or more is Poisson with parameter $\theta(s)(1 - F(s))$, so the unconditional expression for expected surplus is

$$\begin{aligned} \Lambda(s, \theta(s); s) &= s e^{-\theta(s)(1-F(s))} \\ &\quad + e^{-\theta(s)(1-F(s))} \sum_{n=1}^{\infty} \frac{(\theta(s)(1-F(s)))^n}{n!} \left(1 - \int_s^1 \left(\frac{F(x) - F(s)}{1 - F(s)} \right)^n dx \right) \\ &= s e^{-\theta(s)(1-F(s))} + e^{-\theta(s)(1-F(s))} (e^{\theta(s)(1-F(s))} - 1) \\ &\quad - e^{-\theta(s)(1-F(s))} \int_s^1 \sum_{n=1}^{\infty} \frac{(\theta(s)(F(x) - F(s)))^n}{n!} dx; \text{ i.e.,} \\ \Lambda(s, \theta(s); s) &= 1 - \int_s^1 e^{-\theta(s)(1-F(x))} dx. \end{aligned} \tag{8}$$

Next, consider $V(s, \theta(s))$, the expected payoff for a buyer who chooses a seller of type s . Suppose this buyer draws a valuation $x \geq s$. This is the winning draw with probability $e^{-\theta(s)(1-F(x))}$. Conditional on winning, the buyer's payoff is the difference between x and the highest valuation drawn by any other buyers who visited this seller or s if no other buyer drew a valuation $y \geq s$. Now suppose there were n other buyers who drew $y \in [s, x]$. Conditional on n , the winning buyer's expected payoff is then

$$x - \int_s^x y d \left(\frac{F(y) - F(s)}{F(x) - F(s)} \right)^n = \int_s^x \left(\frac{F(y) - F(s)}{F(x) - F(s)} \right)^n dy$$

Summing against the probability mass function for n , the buyer's expected payoff, conditional on winning with valuation $x \geq s$, is

$$\sum_{n=0}^{\infty} \frac{e^{-\theta(s)(F(x)-F(s))} (\theta(s)(F(x) - F(s)))^n}{n!} \int_s^x \left(\frac{F(y) - F(s)}{F(x) - F(s)} \right)^n dy = \int_s^x e^{-\theta(s)(F(x)-F(y))} dy.$$

Multiplying by the probability of winning with valuation x at seller s gives

$$V(s, \theta(s); x) = \int_s^x e^{-\theta(s)(1-F(y))} dy. \tag{9}$$

This is the expected payoff for a buyer who visits seller s conditional on drawing valuation x , where $x \geq s$. Finally, the unconditional expected payoff for a buyer who visits seller s is

$$\begin{aligned} V(s, \theta(s)) &= (1 - F(s)) \int_s^1 V(s, \theta(s); x) \frac{f(x)}{1 - F(s)} dx \\ &= \int_s^1 \int_s^x e^{-\theta(s)(1-F(y))} dy f(x) dx = \int_s^1 (1 - F(x)) e^{-\theta(s)(1-F(x))} dx, \end{aligned} \quad (10)$$

where the last equality follows by integration by parts ($u = \int_s^x e^{-\theta(s)(1-F(y))} dy$, $v = -(1 - F(x))$). With $s = 0$, equation (10) is the expression for expected buyer payoff derived in Peters and Severinov (1997).

Differentiating equation (8) and using equation (10) gives equation (7). This completes the derivation of equation (2).

For completeness, we also derive $\Pi(s, \theta(s); s)$. Consider a seller of type s who posts reserve price equal to s , and suppose this seller attracts n buyers who draw valuations of s or more. The seller gets a payoff of s if $n = 0$ or 1 ; if $n \geq 2$, the seller's expected payoff equals the expected value of the second highest valuation across the buyers, i.e., $E[Y_{n-1}]$. The distribution function of Y_{n-1} is

$$G_{n-1}(y) = \left(\frac{F(y) - F(s)}{1 - F(s)} \right)^n + n \left(\frac{F(y) - F(s)}{1 - F(s)} \right)^{n-1} \left(\frac{1 - F(y)}{1 - F(s)} \right) \text{ for } s \leq y \leq 1.$$

Thus, conditional on $n \geq 2$, the seller's expected payoff is

$$\int_s^1 y dG_{n-1}(y) = 1 - \int_s^1 \left(\frac{F(y) - F(s)}{1 - F(s)} \right)^n dy - n \int_s^1 \left(\frac{(F(y) - F(s))^{n-1} (1 - F(y))}{(1 - F(s))^n} \right) dy.$$

Summing against the probability mass function for n ,

$$\begin{aligned} \Pi(s, \theta(s); s) &= s e^{-\theta(s)(1-F(s))} (1 + \theta(s)(1 - F(s))) + e^{-\theta(s)(1-F(s))} \sum_{n=2}^{\infty} \frac{(\theta(s)(1 - F(s)))^n}{n!} \\ &\quad - e^{-\theta(s)(1-F(s))} \int_s^1 \sum_{n=2}^{\infty} \frac{(\theta(s)(F(y) - F(s))^n}{n!} dy \\ &\quad - e^{-\theta(s)(1-F(s))} \int_s^1 (1 - F(y)) \theta(s) \sum_{n=2}^{\infty} \frac{(\theta(s)(F(y) - F(s)))^{n-1}}{(n-1)!}; \text{ i.e.,} \\ \Pi(s, \theta(s); s) &= 1 - \int_s^1 e^{-\theta(s)(1-F(x))} dx - \int_s^1 \theta(s)(1 - F(x)) e^{-\theta(s)(1-F(x))} dx. \end{aligned} \quad (11)$$

Using equations (8), (10) and (11), it is straightforward to check equation (1). Finally, using

$$1 - \int_s^1 e^{-\theta(s)(1-F(x))} dx = \theta(s) \int_s^1 x e^{-\theta(s)(1-F(x))} f(x) dx,$$

the seller's expected payoff can be rewritten as

$$\Pi(s, \theta(s); s) = \theta(s) \int_s^1 \left(x - \frac{1 - F(x)}{f(x)} \right) e^{-\theta(s)(1-F(x))} f(x) dx.$$

With $s = 0$, this is the expression for expected seller payoff that is given in Peters and Severinov (1997).

A.2 *Ex Ante* Case

In this appendix, we give the details and prove the efficiency of free-entry equilibrium for the *ex ante* case, i.e., the case in which buyers draw their valuations before deciding which seller to visit. As discussed in the text, competition leads sellers to post efficient mechanisms in the *ex ante* case just as it does in the *ex post case*; in particular, a seller with reservation value s posts a second-price auction with reserve price s . However, with a distribution of seller types in the market, the fact that buyers learn their valuations *ex ante* complicates the analysis. The extra complication arises because the distribution of buyers will vary across seller types. The surplus associated with an auction posted by a type s seller should therefore be written as $\Lambda(s, \theta(s), F(x; s); s)$; that is, the surplus depends on the posted reserve price, s , on the expected number of buyers attracted by that reserve price, $\theta(s)$, and on the distribution of valuations across the buyers visiting sellers of type s , $F(x; s)$, as well as on the seller's type, s . Similarly, the expected payoff for seller s is $\Pi(s, \theta(s), F(x; s); s)$, and the expected payoff for a buyer with valuation x who visits a seller posting reserve price s is $V(s, \theta(s), F(x; s); x)$. Applying the approach used in Appendix 1 gives

$$\Lambda(s, \theta(s), F(x; s); s) = 1 - \int_s^1 e^{-\theta(s)(1-F(x;s))} dx \quad (12)$$

$$\Pi(s, \theta(s), F(x; s); s) = \theta(s) \int_s^1 \left(x - \frac{1 - F(x; s)}{f(x; s)} \right) e^{-\theta(s)(1-F(x;s))} f(x; s) dx \quad (13)$$

$$V(s, \theta(s), F(x; s); x) = \int_s^x e^{-\theta(s)(1-F(y;s))} dy \quad (14)$$

For ease of notation, however, we use $\Lambda(s)$, $\Pi(s)$ and $V(s; x)$, respectively. We also simplify the notation by normalizing the measure of buyers, B , to one.

We use the following approach to characterize the equilibrium and the social planner solution in the *ex ante* case. First, we consider the case of two seller types with a mass m_1 of sellers of type s_1 and a mass m_2 of potential sellers of type s_2 , where $0 \leq s_1 < s_2 < 1$. Second, we extend the analysis to the case of N seller types with masses m_1, \dots, m_N

of seller types $0 \leq s_1 < \dots < s_N < 1$. Finally, we move to a continuum of sellers by considering the appropriate limit.

A.2.1 Two Seller Types

We have argued in the text of the paper that equilibrium is characterized by a cutoff, x^* , and a measure, m_2^* , of type s_2 sellers such that

$$V(s_1; x^*) = V(s_2; x^*) \quad (15)$$

$$\Pi(s_2) = A + s_2. \quad (16)$$

Note that x^* is the *lowest* value of x satisfying equation (15); in particular, $V(s_1; x) > V(s_2; x)$ for all $x \in [s_1, x^*)$ while $V(s_1; x) = V(s_2; x)$ for all $x \in [x^*, 1]$.

The corresponding social planner problem is to choose a cutoff, \hat{x} , and a measure, \hat{m}_2 , to maximize

$$m_1 (\Lambda(s_1) - (A + s_1)) + m_2 (\Lambda(s_2) - (A + s_1)).$$

Consider the partial derivative of the social planner maximand with respect to \hat{x} ; in particular, consider an increase in the cutoff from \hat{x} to $\hat{x} + dx$. The key to understanding this derivative is to recognize that the only agents who change their behavior are buyers with valuations in the interval $[\hat{x}, \hat{x} + dx)$. Buyers with valuations $x \in [0, \hat{x})$ randomized over sellers of type s_1 before the change; they continue to do so after the change. Similarly, buyers with valuations $x \in [\hat{x} + dx, 1]$ randomized over all sellers before the change; they continue to do so afterwards.

Buyers with valuations $x \in [\hat{x}, \hat{x} + dx)$ randomized over all sellers before the increase in the cutoff; after the increase, these buyers randomize over sellers of type s_1 . Thus, there are some buyers with valuations in $[\hat{x}, \hat{x} + dx)$ who would have participated in an auction run by a seller of type s_2 before the change but instead participate in an auction run by a seller of type s_1 after the change. To be more precise, approximately $\frac{m_2}{m_1 + m_2} f(\hat{x}) dx$ buyers are expected to switch seller types. When a buyer switches from an auction with reserve price s_2 to one with reserve price s_1 , there is an increase in surplus associated with the auction posted by the type s_1 seller but a decrease in surplus associated with the auction posted by the type s_2 seller. The social planner wants these two effects to offset each other. Were this not the case, e.g., if moving buyers with valuations close to \hat{x} from type s_2 sellers to type s_1 sellers increased the surplus associated with auctions posted by type s_1 sellers more than it decreased the surplus associated with auctions posted by type s_2 sellers, then the social planner should increase the cutoff value.

Consider the reallocation of a buyer with valuation $x \in [\hat{x}, \hat{x} + dx)$ from a type s_2 seller to a type s_1 seller. The increase in surplus at the auction posted by the type s_1 seller is the sum of three components: (i) the increase in the seller's expected payoff, (ii) the decrease in the expected payoffs of any "incumbent" buyers, and (iii) the expected payoff, $V(s_1; x)$, of the buyer who switched seller types. As we have argued in the text, the first two terms are exactly offsetting; thus, the increase in surplus associated with an auction posted by a type s_1 seller that gained a buyer of type x equals $V(s_1; x)$. By the same argument, the decrease in surplus associated with an auction posted by a type s_2 seller that lost a buyer of type x equals $V(s_2; x)$. Now let $dx \rightarrow 0$, so $x \approx \hat{x}$. Satisfying the first-order condition of the social planner problem with respect to the cutoff value requires

$$V(s_1; \hat{x}) = V(s_2; \hat{x}).$$

The cutoff \hat{x} is the lowest value of x satisfying this equation. Equation (15) thus implies $\hat{x} = x^*$; that is, the equilibrium and social planner cutoffs coincide.

Next consider the partial derivative of the social planner maximand with respect to \hat{m}_2 . We can write the first-order condition as

$$m_1 \frac{\partial \Lambda(s_1)}{\partial m_2} + \hat{m}_2 \frac{\partial \Lambda(s_2)}{\partial m_2} + \Lambda(s_2) - (A + s_2) = 0. \quad (17)$$

Since the social planner wants sellers to post efficient mechanisms, we have

$$\Lambda(s_2) = \Pi(s_2) + \theta(s_2)V(s_2),$$

where

$$V(s_2) = \int_{x^*}^1 V(s_2; x) \frac{f(x)}{1 - F(x^*)} dx \quad (18)$$

is a shorthand notation for the expected payoff per buyer visiting a type s_2 seller. Equation (17) implies equation (16) if

$$m_1 \frac{\partial \Lambda(s_1)}{\partial m_2} + \hat{m}_2 \frac{\partial \Lambda(s_2)}{\partial m_2} = -\theta(s_2)V(s_2). \quad (19)$$

We now verify equation (19). Suppose a type s_2 seller enters the market. In expectation, this seller takes $\theta(s_2)$ buyers away from incumbent sellers; thus, $m_1 \frac{\partial \Lambda(s_1)}{\partial m_2} + \hat{m}_2 \frac{\partial \Lambda(s_2)}{\partial m_2}$ is the business-stealing effect associated with the entrant. Any buyer who attempts to purchase the good from the new entrant has a valuation of x^* or more, and these buyers randomize their visits across both seller types. If a buyer with valuation x moves from an incumbent seller to the new entrant, the loss in surplus at the incumbent

seller's auction is $V(s_2; x)$, and since $V(s_1; x) = V(s_2; x)$ for all $x \geq x^*$, i.e., high-valuation buyers are indifferent between the two seller types, this loss is the same irrespective of the type of the incumbent seller. The valuation x is a draw from the truncated density, $\frac{f(x)}{1 - F(x^*)}$; thus, the expected loss to the incumbent seller is $V(s_2)$, as given in equation (18). Multiplying this by the expected number of buyers who visit the new entrant gives equation (19).

A.2.2 N Seller Types

Suppose there are N seller types, $0 \leq s_1 < \dots < s_N < 1$, with respective measures m_1, \dots, m_N , where we consider the entry decision of the marginal seller type, s_N . In equilibrium, there will exist $N - 1$ thresholds $x^*(s_2), \dots, x^*(s_N)$ such that buyer types in $[x^*(s_k), x^*(s_{k+1})]$ randomize among sellers s_1, \dots, s_k . Buyers of type $x < s_1$ do not participate in the market; that is, $x^*(s_1) = s_1$. Equilibrium is characterized by the cutoffs, $x^*(s_1), x^*(s_2), \dots, x^*(s_N)$, and a measure of sellers of type s_N such that

$$\begin{aligned} 0 &= V(s_1; x^*(s_1)) \\ V(s_1; x^*(s_2)) &= V(s_2; x^*(s_2)) \end{aligned} \tag{20}$$

$$\begin{aligned} V(s_1; x^*(s_3)) &= V(s_2; x^*(s_3)) = V(s_3; x^*(s_3)) \\ &\dots \\ V(s_1; x^*(s_k)) &= V(s_2; x^*(s_k)) = \dots = V(s_k; x^*(s_k)) \end{aligned} \tag{21}$$

$$\begin{aligned} &\dots \\ V(s_1; x^*(s_N)) &= V(s_2; x^*(s_N)) = \dots = V(s_N; x^*(s_N)) \\ \Pi(s_N) &= A + s_N. \end{aligned} \tag{22}$$

The corresponding social planner problem is to choose cutoffs, $\hat{x}(s_1), \hat{x}(s_2), \dots, \hat{x}(s_N)$, and a measure, \hat{m}_N , of type s_N sellers to maximize

$$\sum_{k=1}^N m_k (\Lambda(s_k) - (A + s_k)).$$

The proof that $\hat{x}(s_k) = x^*(s_k)$ for $k = 1, \dots, N$ is essentially the same as the one given for the case of two seller types.

First, it is obvious that $x^*(s_1) = \hat{x}(s_1) = s_1$. Buyers with $x \leq s_1$ have no incentive to participate in the market nor does the social planner want them to do so. Second, given any collection of cutoffs for sellers of type s_3 and above and given any level of entry by type s_N sellers, the choice of $\hat{x}(s_2)$ does not affect $\sum_{k=3}^N m_k (\Lambda(s_k) - (A + s_k))$. The

same argument that was used to characterize $\hat{x}(s_2)$ in the two-seller case then implies that the social planner sets $\hat{x}(s_2)$ so that

$$V(s_1; \hat{x}(s_2)) = V(s_2; \hat{x}(s_2)).$$

Comparing this with equation (20) gives $\hat{x}(s_2) = x^*(s_2)$. The final step in the argument uses induction. Suppose $\hat{x}(s_i) = x^*(s_i)$ for $i = 1, \dots, k-1$, and take $\{\hat{x}(s_{k+1}), \dots, \hat{x}(s_N), m_N\}$ as given. The choice of $\hat{x}(s_k)$ has no effect on $\sum_{i=k+1}^N m_i (\Lambda(s_i) - (A + s_i))$. Since $\{\hat{x}(s_1), \hat{x}(s_2), \dots, \hat{x}(s_{k-1})\}$ are assumed to have been set optimally, the social planner is indifferent between assigning the buyer with valuation $\hat{x}(s_k)$ to seller s_k versus assigning that buyer to any seller with a lower reservation value. That is,

$$V(s_1; \hat{x}(s_k)) = V(s_2; \hat{x}(s_k)) = \dots = V(s_k; \hat{x}(s_k));$$

thus, by comparison with equation (21), $\hat{x}(s_k) = x^*(s_k)$. By induction, the equilibrium and social planner cutoff values coincide for $i = 1, \dots, N$.

Finally, in order that $\hat{m}_N = m_N^*$, it must be that

$$\sum_{k=1}^N m_k \frac{\partial \Lambda(s_k)}{\partial m_M} = -\theta(s_N) V(s_N);$$

that is, the business-stealing effect associated with the entry of a type s_N seller has to equal the expected number of buyers drawn away by the entry of the marginal seller times the loss in surplus per buyer who leaves an incumbent's auction. The argument for why this equation holds is exactly the same as in the case with two seller types.

A.2.3 Continuum of Seller Types

In the model with N seller types, for each seller type s_k , there is a corresponding buyer type $x^*(s_k)$ who is indifferent between visiting a seller of type s_k versus any seller posting a lower reserve price. The function $x^*(s_k)$ is defined on a discrete set of points, $\{s_1, \dots, s_N\}$. To move to a continuum of seller types, we let the distance between seller types s_{k+1} and s_k go to zero and derive a differential equation that gives a continuous function $x^*(s)$ as the limit of the N -seller case. The purpose of this subsection is to derive this equation. Since the continuum-of-seller-types solution is the limit of the discrete seller type case, our efficiency results carry over to the continuum.

As in the *ex post* case, we normalize the total measure of potential sellers to one, and we denote the distribution of reservation values across these seller by $G(s)$. We begin with a discrete distribution over seller types. Let $s_1 = 0$, $s_2 = \Delta s, \dots, s_{k+1} = s_k + \Delta s$, and

let $m_1 = g(s_1)\Delta s$, $m_2 = g(s_2)\Delta s$, etc. We denote the arrival rate of buyers to type s_k sellers by $\theta(s_k)$ and the distribution of valuations among buyers visiting type s_k sellers by $F(x; s_k)$.

Lemma 1

$$\theta(s_k) = \theta(s_{k+1}) + \frac{1}{\sum_{j=1}^k m_j} \int_{x^*(s_k)}^{x^*(s_{k+1})} f(x) dx \text{ for } k = 1, \dots, N-1 \quad (23)$$

$$\theta(s_k)F(x; s_k) = \frac{1}{\sum_{j=1}^k m_j} \int_{x^*(s_k)}^x f(z) dz \text{ for } x^*(s_k) \leq x \leq x^*(s_{k+1}). \quad (24)$$

Proof. Buyers with valuations $x \geq x^*(s_{k+1})$ randomize across all sellers of type s_{k+1} or below. Thus, a type s_k seller can expect as many buyers of this type as can a type s_{k+1} seller. In addition, a type s_k seller attracts some additional buyers, namely, those with valuations $x \in [x^*(s_k), x^*(s_{k+1}))$. Buyers with valuations in this range randomize over sellers of type s_k and below, and there is a mass $\sum_{j=1}^k m_j$ of such sellers. This gives equation (23). To understand equation (24), note that (i) the measure of the buyers with valuations between $x^*(s_k)$ and $x < x^*(s_{k+1})$ visiting type s_k sellers is $\left(\frac{m_k}{m_1 + \dots + m_k}\right) \int_{x^*(s_k)}^x f(z) dz$ while (ii) the measure of type s_k sellers can be written as $m_k \theta(s_k)$. Since $F(x; s_k) = 0$ for $x \leq x^*(s_k)$, it follows that

$$F(x; s_k) = \frac{\left(\frac{m_k}{m_1 + \dots + m_k}\right) \int_{x^*(s_k)}^x f(z) dz}{m_k \theta(s_k)} \text{ for } x^*(s_k) \leq x \leq x^*(s_{k+1}).$$

Multiplying both sides by $\theta(s_k)$ gives equation (24). ■

The cutoff valuation $x^*(s_{k+1})$, i.e., the lowest buyer type who is indifferent between visiting a type s_k seller versus a type s_{k+1} seller, is defined by $V(s_k; x^*(s_{k+1})) = V(s_{k+1}; x^*(s_{k+1}))$. Using equation (14) gives

$$\int_{s_k}^{x^*(s_{k+1})} e^{-\theta(s_k)(1-F(x; s_k))} dx = \int_{s_{k+1}}^{x^*(s_{k+1})} e^{-\theta(s_{k+1})(1-F(x; s_{k+1}))} dx. \quad (25)$$

Note that $F(x; s_k) = 0$ for $x < x^*(s_k)$, and similarly $F(x; s_{k+1}) = 0$ for $x < x^*(s_{k+1})$. Equation (25) can thus be rewritten as

$$\begin{aligned} e^{-\theta(s_k)} (x^*(s_k) - s_k) + \int_{x^*(s_k)}^{x^*(s_{k+1})} e^{-\theta(s_k)(1-F(x; s_k))} dx &= e^{-\theta(s_{k+1})} (x^*(s_{k+1}) - s_{k+1}) \text{ or} \\ x^*(s_k) - s_k + \int_{x^*(s_k)}^{x^*(s_{k+1})} e^{\theta(s_k)F(x; s_k)} dx &= e^{\theta(s_k) - \theta(s_{k+1})} (x^*(s_{k+1}) - s_{k+1}) \end{aligned}$$

Using

$$e^{\theta(s_k)F(x;s_k)} \simeq 1 + \theta(s_k)F(x;s_k) \text{ and } e^{\theta(s_k)-\theta(s_{k+1})} \simeq 1 + \theta(s_k) - \theta(s_{k+1})$$

equation (25) can be further rewritten as

$$x^*(s_{k+1}) - s_k + \int_{x^*(s_k)}^{x^*(s_{k+1})} \theta(s_k)F(x;s_k)dx = (1 + \theta(s_k) - \theta(s_{k+1})) (x^*(s_{k+1}) - s_{k+1}). \quad (26)$$

We use the notation $s_{k+1} = s$, $s_k = s - \Delta s$, $x^*(s_{k+1}) = x^*(s)$, and $x^*(s_k) = x^*(s) - \Delta x^*(s)$ and note that $G(s) = \sum_{j=1}^k m_j$. Then using equations (23) and (24), we can rewrite equation (26) as

$$x^*(s) - s + \Delta s + \int_{x^*(s) - \Delta x^*(s)}^x \int_{x^*(s) - \Delta x^*(s)}^x \frac{f(z)dz}{G(s)} dx = (x^*(s) - s) \left(1 + \frac{\int_{x^*(s) - \Delta x^*(s)}^{x^*(s)} f(x)dx}{G(s)} \right). \quad (27)$$

The left-hand side of equation (27) used equation (24); the right-hand side used equation (23). The term $\int_{x^*(s) - \Delta x^*(s)}^x \int_{x^*(s) - \Delta x^*(s)}^x \frac{f(z)dz}{G(s)} dx$ on the left-hand side of this equation is $o(\Delta x^*(s))$. On the right-hand side,

$$\int_{x^*(s) - \Delta x^*(s)}^{x^*(s)} f(x)dx = F(x^*(s)) - F(x^*(s) - \Delta x^*(s)) \simeq f(x^*(s))\Delta x^*(s).$$

Equation (27) therefore reduces to

$$\frac{\Delta x^*(s)}{\Delta s} = \frac{G(s)}{(x^*(s) - s) f(x)}. \quad (28)$$

Together with the initial condition, $x^*(0) = 0$, equation (28) determines the function $x^*(s)$.